APPLICATION OF DERIVATIVES-MODULE 7

Question

Find the maximum and minimum values, if any, of the following given by

$f(x) = (2x-1)^2 + 3$

Solution 1:

The given function is $f(x) = (2x-1)^2 + 3$

It can be observed that $(2x-1)^2 \geq 0$ for every $x \in \mathbf{R}$.

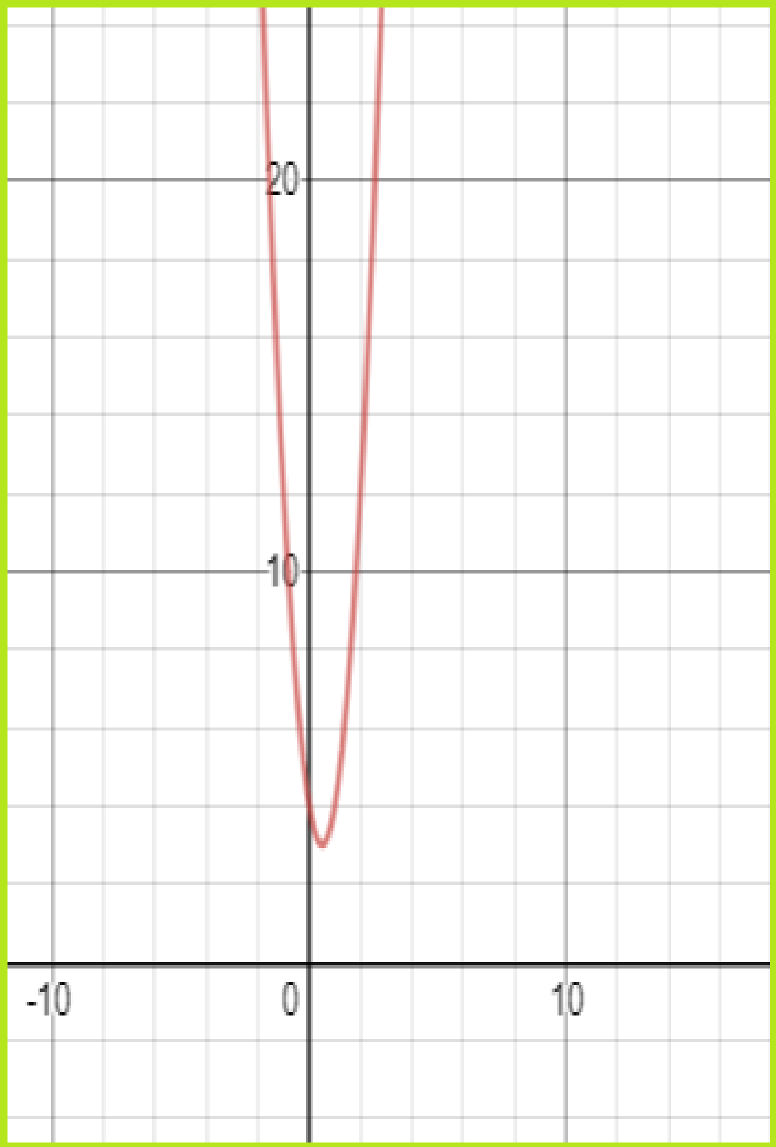
Therefore, $f(x) = (2x-1)^2 + 3 \geq 3$ for every $x \in \mathbf{R}$.

The minimum value of f is attained when $2x - 1 = 0$.

$2x - 1 = 0$, $x = \frac{1}{2}$

Minimum value of $f\left(\frac{1}{2}\right) = \left(2.\frac{1}{2} - 1\right)^2 + 3 = 3$.

Hence, function f does not have a maximum value.



Question

Find the maximum and minimum values, if any, of the following given by

$$f(x) = -(x-1)^2 + 10$$

The given function is $f(x) = -(x-1)^2 + 10$.

It can be observed that $(x-1)^2 \geq 0$ for every $x \in \mathbf{R}$.

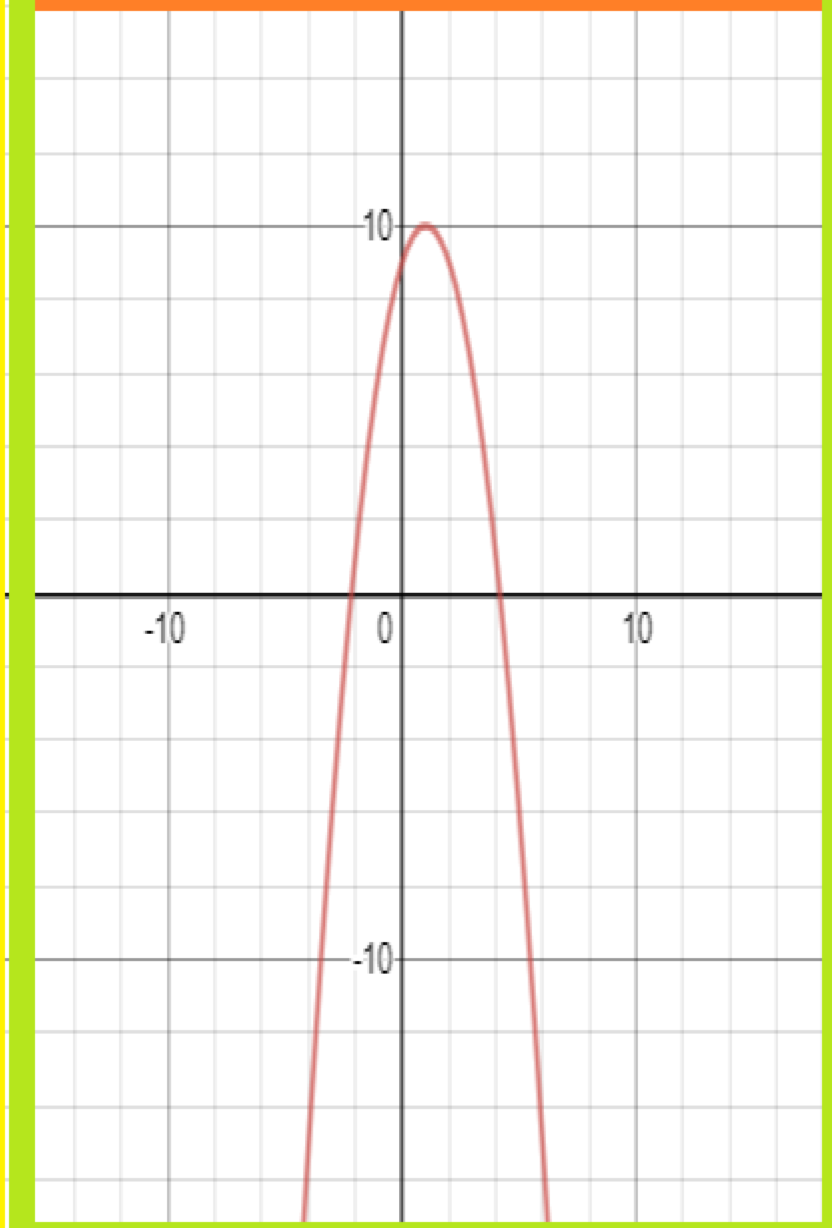
Therefore, $f(x) = -(x-1)^2 + 10 \leq 10$ for every $x \in \mathbf{R}$.

The maximum value of f is attained when $(x-1) = 0$.

$(x-1) = 0$, $x = 0$

Maximum value of $f = f(1) = -(1-1)^2 + 10 = 10$

Hence, function f does not have a minimum value.



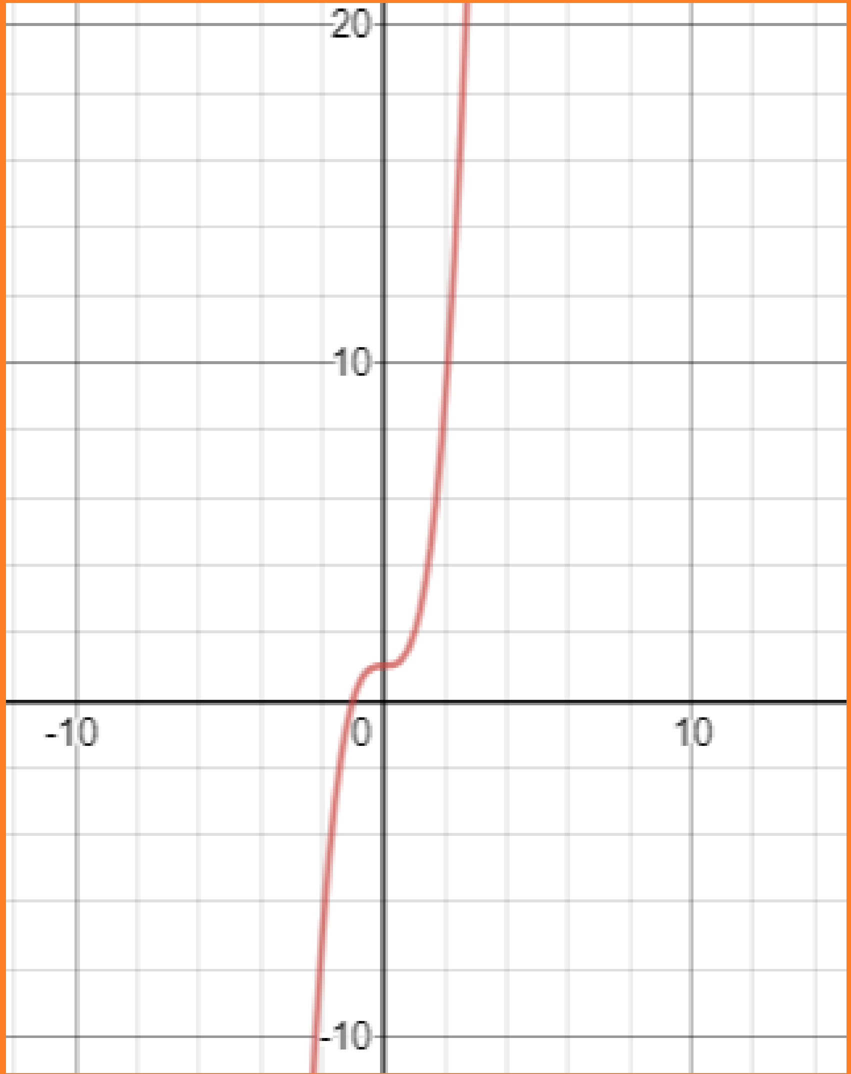
Question

Find the maximum and minimum values, if any, of the following given by

$$g(x) = x^3 + 1$$

The given function is $g(x) = x^3 + 1$.

Hence, function g neither has a maximum value nor a minimum value.



LET US CONCLUDE

Definition 1: Let f be a function defined on an interval I. Then

(a) f is said to have a maximum value in I, if there exists a point c in I such that $f(c) \geq f(x)$, for all $x \in I$.

The number f(c) is called the maximum value of f in I and the point c is called a point of maximum value of f in I.

(b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) \leq f(x)$, for all $x \in I$.

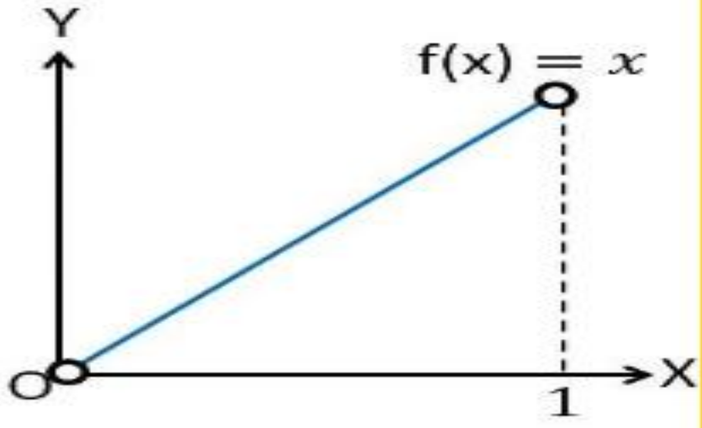
The number f(c), in this case, is called the minimum value of f in I and the point c, in this case, is called a point of minimum value of f in I.

(c) f is said to have an extreme value in I if there exists a point c in I such that f(c) is either a maximum value or a minimum value of f in I.

The number f(c), in this case, is called an extreme value of f in I and the point c is called an extreme point.

Find the maximum and the minimum values, if any, of the function given by $f(x) = x, x \in (0, 1)$.

We first draw graph of $f(x) = x, \quad x \in (0,1)$



Here, points 0 and 1 are not included

f will have

Minimum value of point closest to 0

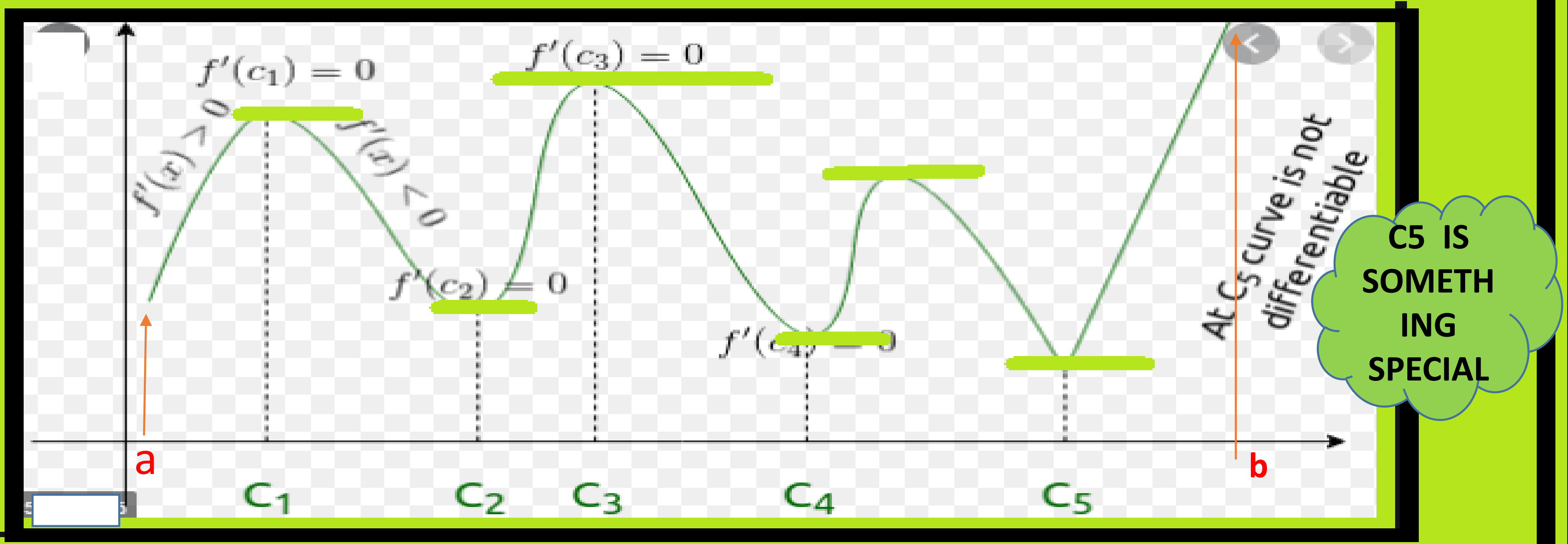
& Maximum value of point closest to 1

but its not possible to locate such points

Thus **the given function has neither Maximum value nor Minimum value**

- THEN THE FUNCTION HAS MINIMUM VALUE 0, AT X = 0 AND MAXIMUM VALUE 1, X = 1
- EVERY MONOTONIC FUNCTION ASSUMES ITS MAXIMUM/MINIMUM VALUE AT THE END POINTS OF THE DOMAIN OF THE FUNCTION
- EVERY CONTINUOUS FUNCTION ON A CLOSED INTERVAL HAS A MAXIMUM AND A MINIMUM VALUE
- A MONOTONIC FUNCTION f in the interval I, MEANS f IS EITHER INCREASING IN I OR DECREASING IN I

POINT OF LOCAL MAXIMA & LOCAL MINIMA



Definition 2: Let f be a real valued function and let c be an interior point in the domain of f. Then

(a) c is called a point of local maxima if there is an $h > 0$ such that $f(c) \geq f(x)$, for all x in $(c - h, c + h)$

The value f(c) is called the local maximum value of f.

(b) c is called a point of local minima if there is an $h > 0$ such that $f(c) \leq f(x)$, for all x in $(c - h, c + h)$

The value f(c) is called the local minimum value of f.

The above definition leads to the following theorem:

Theorem: Let f be a function defined on an open interval I. Suppose $c \in I$ be any point. If f has a local maxima or a local minima at $x = c$, then either $f'(c) = 0$ or f is not differentiable at c.

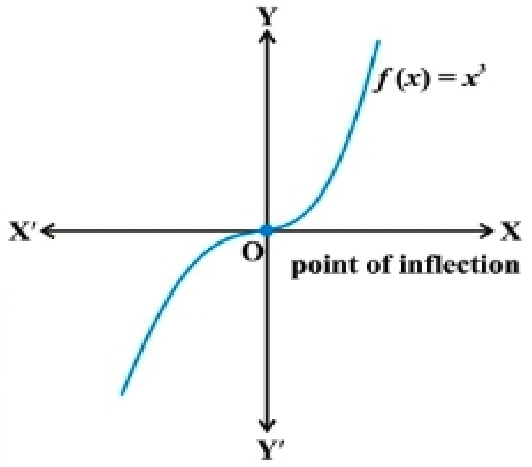
Now, using only the first order derivatives, we calculate for finding points of local maxima or points of local minima.

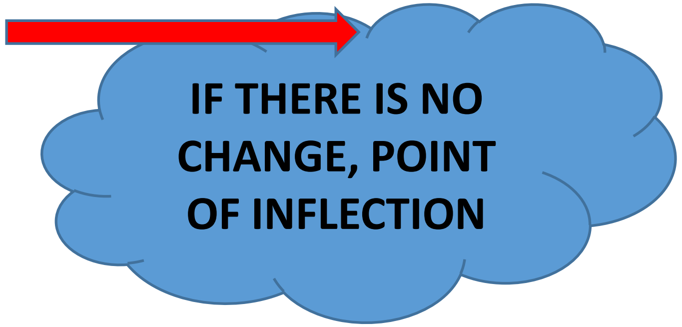
Theorem (First Derivative Test): Let f be a function defined on an open interval I. Again let f be continuous at a critical point c in I. Then

(i) If f '(x) changes sign from positive to negative as x increases through c, i.e. if f '(x) > 0 at every point sufficiently close to and to the left of c, and f '(x) < 0 at every point sufficiently close to and to the right of c, then c is a point of local maxima.

(ii) If f '(x) changes sign from negative to positive as x increases through c, i.e. if f '(x) < 0 at every point sufficiently close to and to the left of c, and f '(x) > 0 at every point sufficiently close to and to the right of c, then c is a point of local minima.

(iii) If f '(x) does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Such a point is called point of inflection as shown in the figure.





POINT OF LOCAL MINIMA

$f(x) = (x - 5)^4$

Solution:

Given $f(x) = (x - 5)^4$

Differentiate with respect to x

$f'(x) = 4(x - 5)^3$

For local maxima and minima

$f'(x) = 0$

$= 4(x - 5)^3 = 0$

$= x - 5 = 0$

$x = 5$

X = 5 is the only one critical point

VALUE OF X	SIGN OF $f'(x)=4(x-5)^3$	
x = 4.9	$4((4.9 - 5)^3$ = -0.004	-ve
X = 5.1	$4((5.1 - 5)^3$ = 0.004	+ve

THERE FORE X = 5 IS THE POINT OF LOCAL MINIMA.

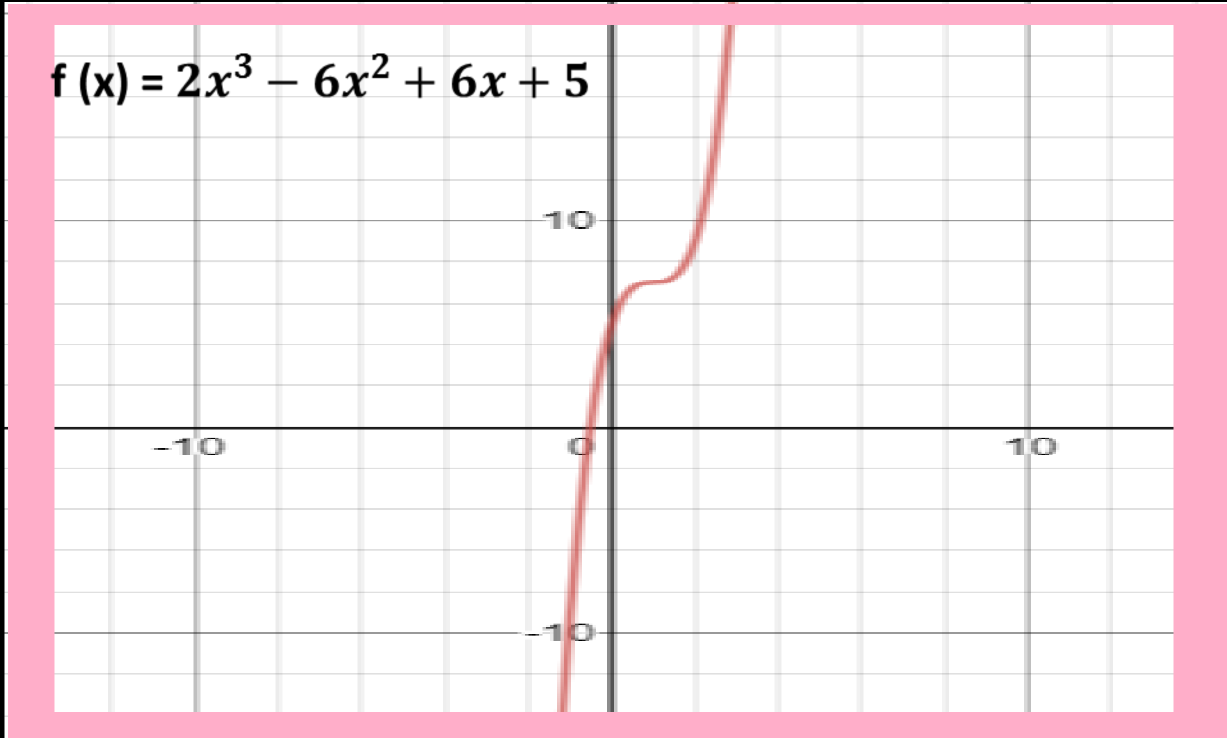
FIND ALL POINTS OF LOCAL MAXIMA AND MINIMA OF THE FUNCTION GIVEN BY

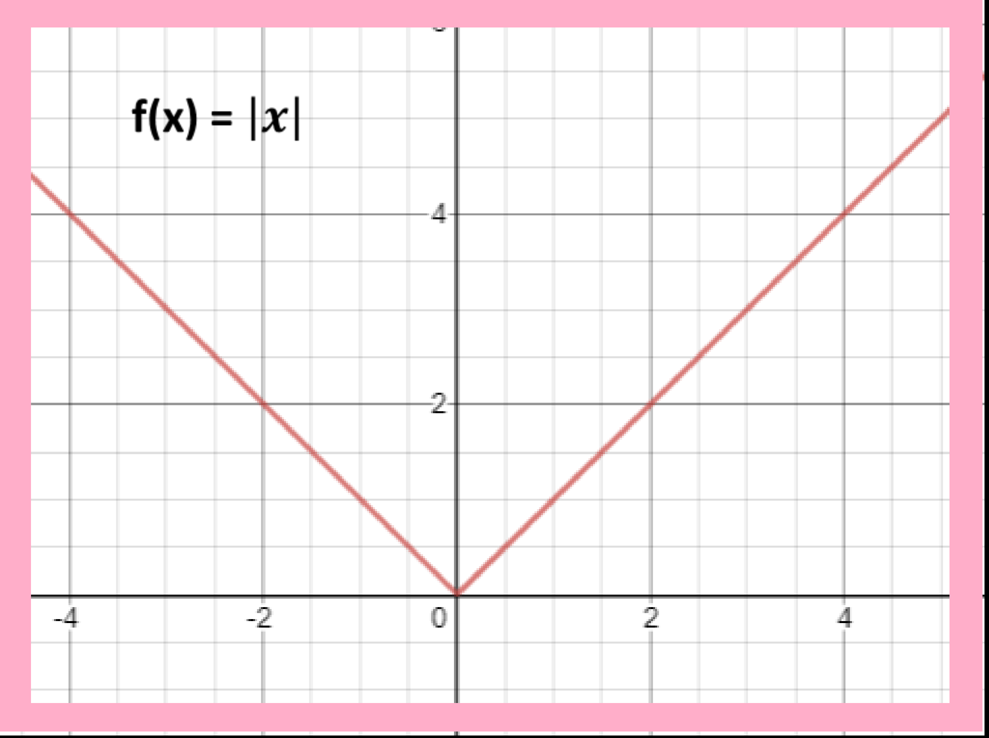
- f (x) = $2x^3 - 6x^2 + 6x + 5$
- $f'(x) =$ $6x^2$-12x + 6 = $6(x-1)^2$
- If $f'(x) = 0$, gives x = 1. x = 1 is the only critical point

VALUE OF X	Sign of $f'(x)$	
X = 0.9	$6(0.9-1)^2$ = + ve	+ve TO +ve
X = 1.1	$6\ (1.1-1)^2$ = + ve	NO CHANGE

THERE FORE BY FIRST DERIVATIVE TEST X = 1 IS NEITHER A POINT OF LOCAL MAXIMA NOR A POINT OF LOCAL MINIMA.HENCE X = 1 IS A POINT OF INFLEXION

CRITICAL POINT



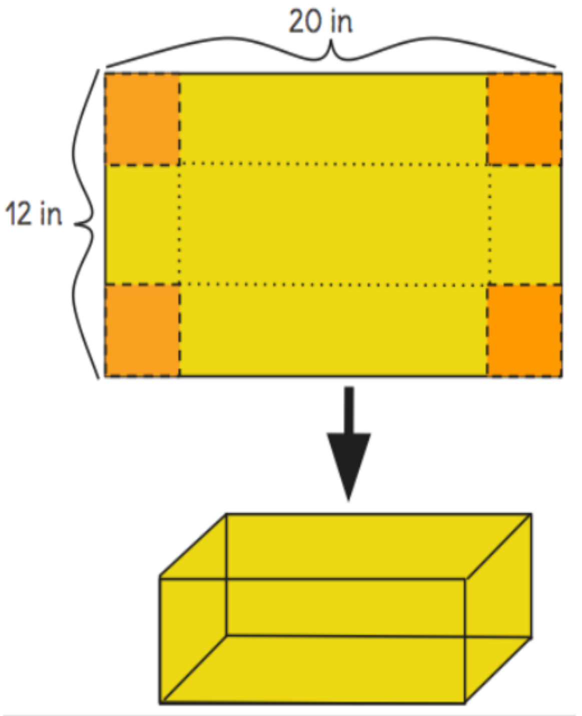


AT A POINT C IN THE DOMAIN OF THE FUNCTION , AT WHICH EITHER $f'(c) = 0$ *or f is not differentiable is called a* CRITICAL POINT OF f

APPLICATION OF DERIVATIVES-MODULE 8

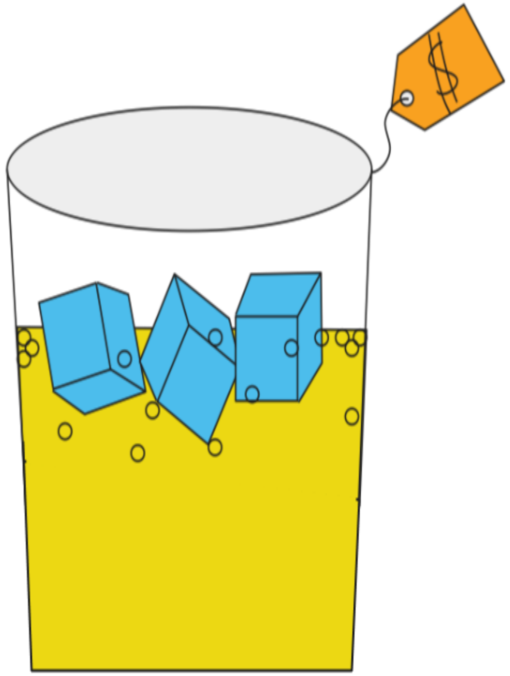
Task

Jerry wants to make a box out of a sheet of cardboard by cutting out congruent squares from each of the four corners and folding up the remaining sides. If the cardboard sheet is 12 inches by 20 inches, what is the maximum volume that can be obtained by doing this?

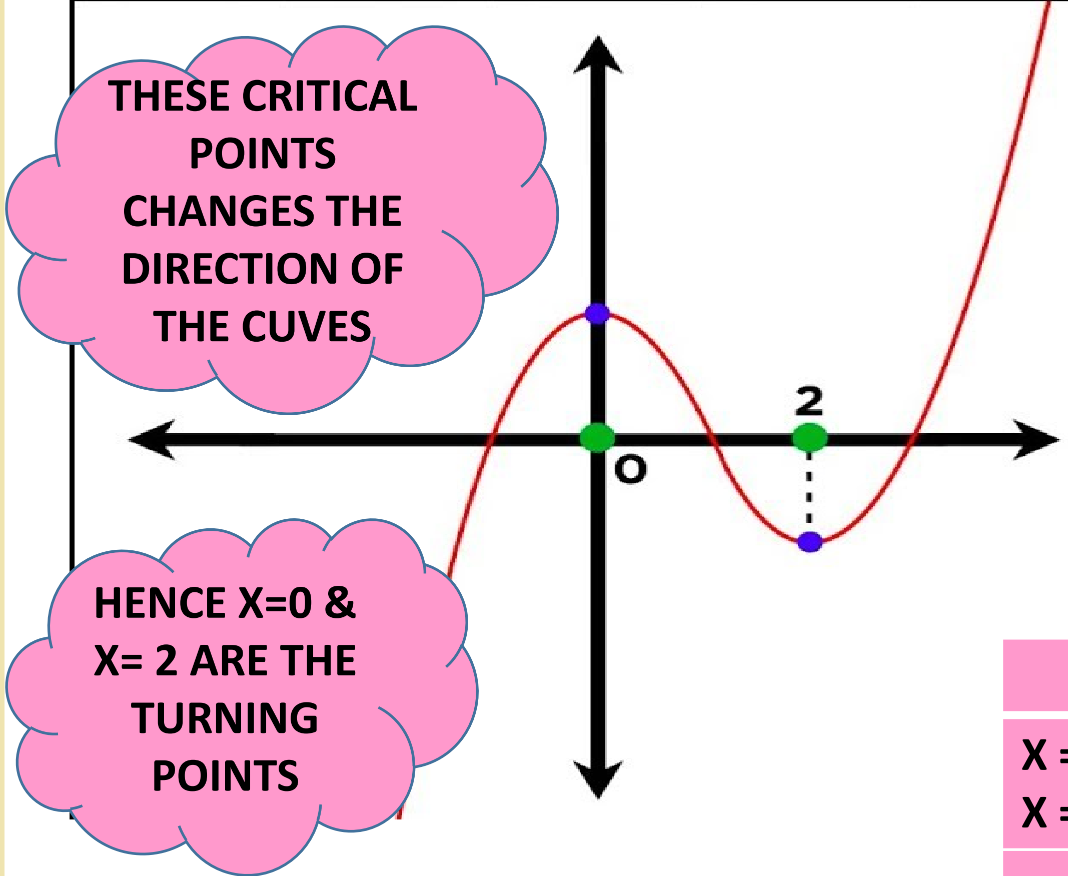


Task

Bobby and Sarah want to sell lemonade at a lemonade stand. Last year they sold 200 cups when they charged $1.00 per cup. Other vendors with similar sales have noted that for every $0.25 increase in price, they sold 20 cups less. Using this information, what should they charge to maximize their income?



Finding Local Maxima and Minima By 1st Derivative Test



$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x$$

$$f'(x) = 3x(x - 2)$$

$$3x(x - 2) = 0$$

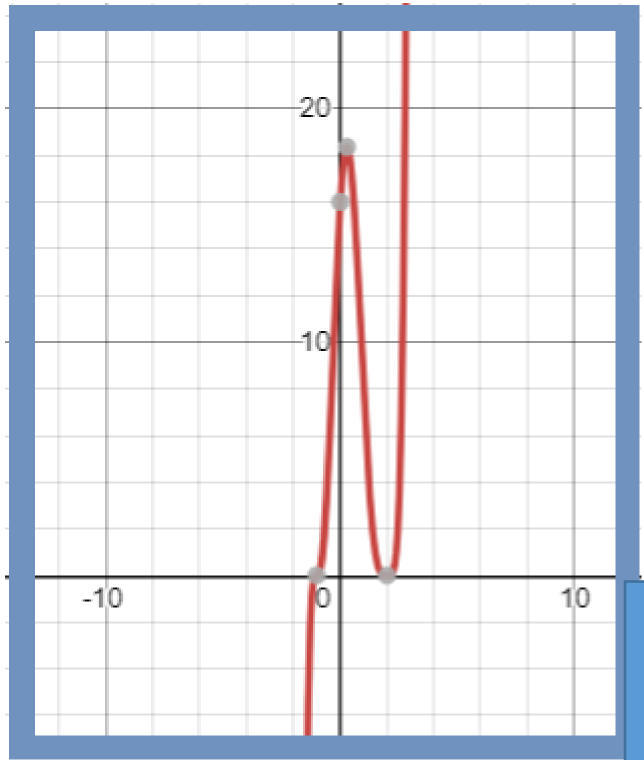
$$f'(x) = 0 \Rightarrow x = 0, \quad x = 2$$

CRITICAL POINTS

	SIGN Of f'(x)	changes
X = -0.1 X = 0. 1	+ -	+ ve to -ve
X = 1.9 X= 2.1	- +	-ve to +ve

THERE FORE BY 1ST DER.TEST, X = 0 IS A POINT OF LOCAL MAXIMA & X = 2 IS A POINT OF LOCAL MINIMA

Find the points at which the function *f* given by $f(x) = (x-2)^4(x+1)^3$ has
(i) local maxima **(ii)** local minima **(iii)** point of inflexion



Solution

The given function is $f(x) = (x-2)^4(x+1)^3$

$$f'(x) = 4(x-2)^3(x+1)^3 + 3(x+1)^2(x-2)^4$$
$$= (x-2)^3(x+1)^2[4(x+1)+3(x-2)]$$
$$= (x-2)^3(x+1)^2(7x-2)$$

Now, $f'(x) = 0 \Rightarrow x = -1$ and $x = \frac{2}{7}$ or $x = 2$

X = -1.1 X = -0.9	(-)(+)(-) = +ve (-)(+)(-) =+ve
X = 1.9/7 X = 2.1/7	(-)(+)(-) = +ve (-)(+)(+) = -ve
X = 1.9 X = 2.1	(-)(+)(+) = - ve (+)(+)(+) = +ve

1) X = 2/7 is a point of local maxima.
2) X = 2 is a point of local minima.
3) X = - 1 is a point of inflexion

IS THIS TOO LENGTHY

Theorem (Second Derivative Test): Let f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then

(i) x = c is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

The value f (c) is local maximum value of f.

(ii) x = c is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$

In this case, f (c) is local minimum value of f.

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.

IS 1ST DERIVATIVE METHOD TOO LENGTHY FOR COMPLICATED FUNCTIONS???

Find local maximum and local minimum values of the function f given by $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$

$f'(x) = 12x^3 + 12x^2 - 24x + 0$

$f'(x) = 12(x^3 + x^2 - 2x)$

$f'(x) = 12x(x-1)(x+2)$

for max/min $f'(x) = 0$

So, $x = 0, x = 1, \& x = -2$ are the critical point

$f''(x) = 12(3x^2 + 2x - 2)$

f''(0) = -24 < 0 ,x = 0 is a point of local maxima
f"(1) = 36 > 0 , x = 1 is a point of local minima
f"(-2) = 73 > 0 , x - -2 is a point of local minima

Finding local minimum and maximum value

Local maximum value of $f(x)$ at $x = 0$

$f(0) = 3(0)^4 + 4(0)^3 - 12(0)^2 + 12 = 0 + 0 - 0 + 12 = \mathbf{12}$

Local minimum value of $f(x)$ at $x = 1$

$f(1) = 3(1)^4 + 4(1)^3 - 12(1)^2 + 12 = 3 + 4 - 12 + 12 = \mathbf{7}$

Local Minimum value of $f(x)$ at $x = -2$

$f(-2) = 3(-2)^4 + 4(-2)^3 - 12(-2)^2 + 12$

$= 48 - 32 - 48 + 12 = \mathbf{-20}$

Find all the points of local maxima and local minima of the function

f given by $f(x) = 2x^3 - 6x^2 + 6x + 5$.

$$f(x) = 2x^3 - 6x^2 + 6x + 5$$

$$f'(x) = 6x^2 - 12x + 6 + +0$$

$$f'(x) = 6(x^2 - 2x + 1)$$

$$f'(x) = 6(x-1)^2$$

Putting $f'(x) = 0$

$$6(x-1)^2 = 0$$

$x = 1$ is the critical point

$$f''(x) = 6 \times 2(x-1)$$

$$f''(x) = 12(x-1)$$

Putting $x = 1$

$$f''(x) = 12(1-1)$$

$$= 0$$

$\therefore$ Second derivative test fails

we will go back to first derivative test

VALUE OF X	Sign of $f'(x)$	
X = 0.9	$6(0.9-1)^2$ = + ve	+ve TO +ve
X = 1.1	$6(1.1-1)^2$ = + ve	NO CHANGE

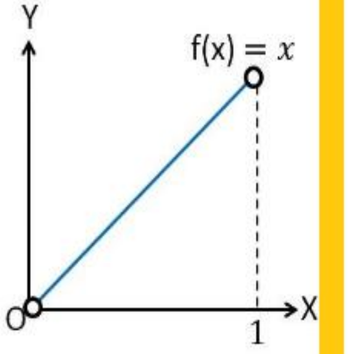
$\Rightarrow x = 1$ is neither point of maxima nor point of minima

Thus, $x = 1$ is **point of inflexion**

MAX/MIN VALUES I N A CLOSED INTERVAL

Find the maximum and the minimum values, if any, of the function given by $f(x) = x, x \in (0, 1)$.

We first draw graph of $f(x) = x, \ x \in (0,1)$



Here, points 0 and 1 are not included

f will have

Minimum value of point closest to 0

& Maximum value of point closest to 1

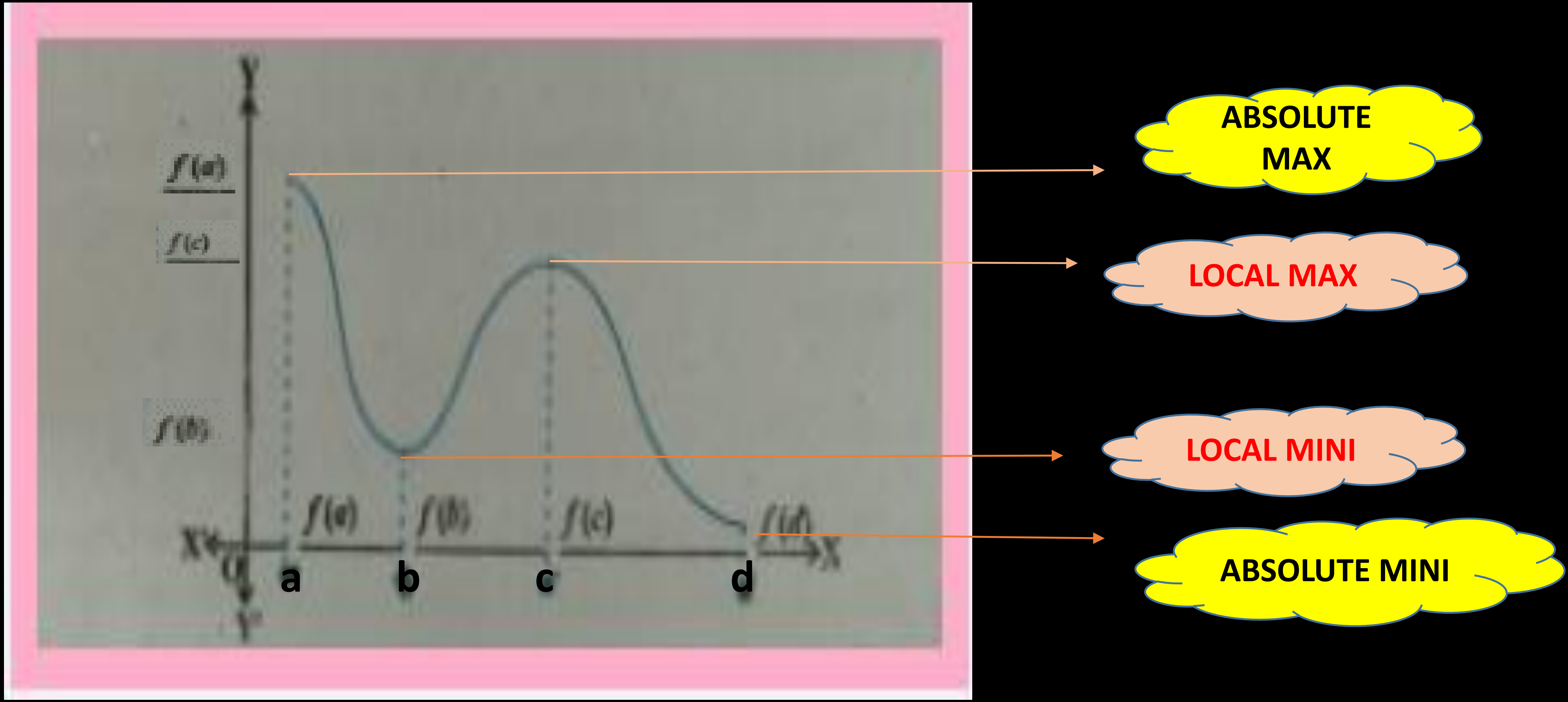
but its not possible to locate such points

Thus **the given function has neither Maximum value nor Minimum value**

- **IF WE CHANGE THE INTERVAL TO [0,1]**

- ❖ **THEN THE FUNCTION HAS MINIMUM VALUE 0, AT X = 0 AND MAXIMUM VALUE 1, X = 1**
- ❖ **THIS MAXIMUM VALUE 1 OF f AT X = 1 IS CALLED THE ABSOLUTE MAXIMUM VALUE(GLOBAL MAXIMUM OR GREATEST VALUE) OF ON THE INTERVAL [0,1]**
- ❖ **THIS MINIMUM VALUE 0 IS CALLED THE ABSOLUTE MINIMUM VALUE (GLOBAL MINIMUM OR LEAST VALUE)**

ABSOLUTE MAXIMUM /MINIMUM VALUES



WORKING RULE

- 1) FIND ALL CRITICAL POINTS OF f IN THE GIVEN INTERVAL(ie., either

  f'(x) = 0 or not differentiable ,eg: $c_{1,} c_2$....)
- 2)TAKE END POINTS OF THE CLOSED INTERVAL [a,b]
- 3) CALCULATE THE VALUE OF THE FUNCTION AT ALL THESE POINTS(eg: f(a), f($c_{1,}$) , f($c_{2,}$) ,....f(b))
- 4) IDENTIFY THE MAXIMUM AND MINIMUM VALUES OF THE FUNCTION.THIS MAXIMUM VALUE WILL BE THE ABSOLUTE MAXIMUM VALUE OF f, AND MINIMUM VALUE WILL BE THE ABSOLUTE MINIMUM VALUE OF THE FUNCTION

Find the absolute maximum and minimum values of a function f given by $f(x) = 2x^3 - 15x^2 + 36x + 1$ on the interval [1, 5].

$f(x) = 2x^3 - 15x^2 + 36x + 1$

$f'(x) = 6x^2 - 30x + 36$

For max/min $f'(x) = 0$

$$6x^2 - 30x + 36 = 0$$

$$6(x-2)(x-3) = 0$$

Hence $x = 2$ or $x = 3$ are the critical points

We are given interval [1 ,5] , end points are 1,5

Hence calculating $f(x)$ at $x = 2, 3, 1, 5$

Value of x	Value of $f(x) = 2x^3 - 15x^2 + 36x + 1$	
1	$f(x) = 2x^3 - 15x^2 + 36x + 1 = \mathbf{24}$	←Minimum
2	$f(x) = 2 \times 2^3 - 15 \times 2^2 + 36 \times 2 + 1 = 29$	
3	$f(x) = 2 \times 3^3 - 15 \times 3^2 + 36 \times 3 + 1 = 28$	
5	$f(x) = 2 \times 5^3 - 15 \times 5^2 + 36 \times 5 + 1 = \mathbf{56}$	←Maximum

Hence,

Absolute maximum value is 56 at $x = 5$

Absolute minimum value is 24 at $x = 1$

Find absolute maximum and minimum values of a function f given by $f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}}, x \in [-1, 1]$

$f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}}$

$f'(x) = 12 \times \frac{4}{3}x^{\frac{4}{3}-1} - 6 \times \frac{1}{3}x^{\frac{1}{3}-1}$

$= 16x^{\frac{1}{3}} - 2x^{\frac{-2}{3}}$

$= \frac{2(8x-1)}{x^{\frac{2}{3}}} = 0$

Putting $f'(x) = 0$; $\frac{2(8x-1)}{x^{\frac{2}{3}}} = 0$

$x = \frac{1}{8}$

Since $f'(x) = \frac{2(8x-1)}{x^{\frac{2}{3}}}$

$f'(x)$ is not defined at $x = 0$

$x = \frac{1}{8}$ & 0 are critical points

We are given interval $[-1, 1]$

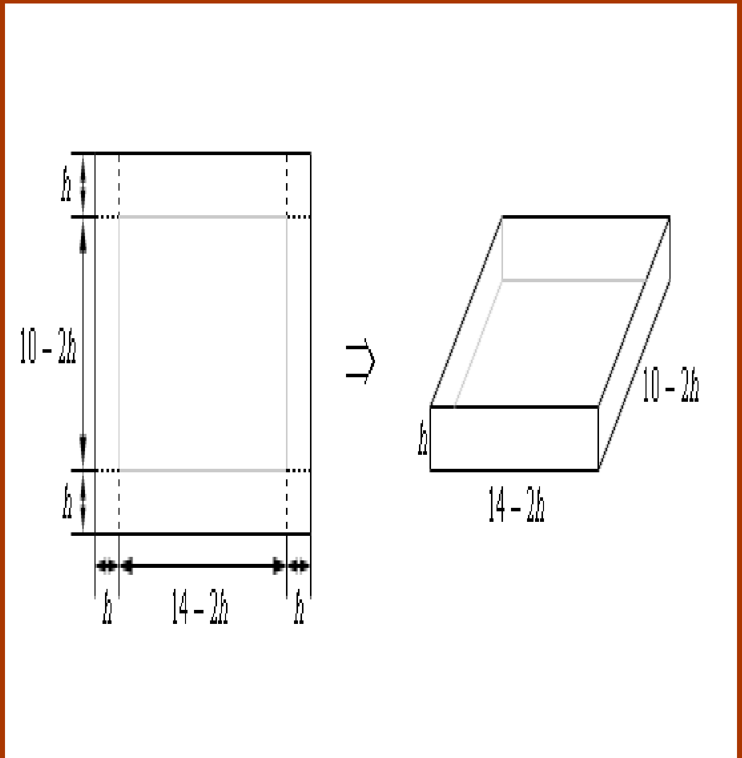
Hence calculating $f(x)$ at $x = 0, \frac{1}{8}, -1, 1$

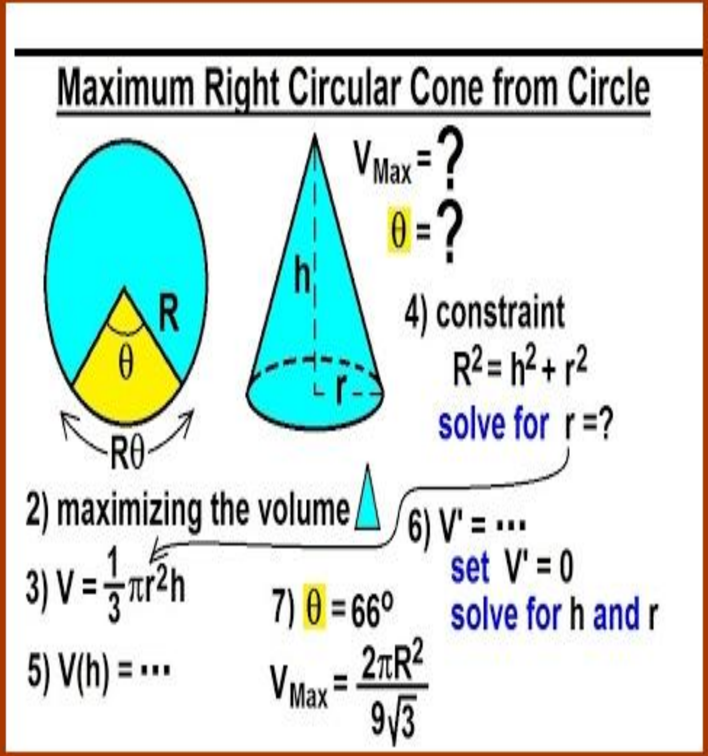
We are given interval $[-1, 1]$

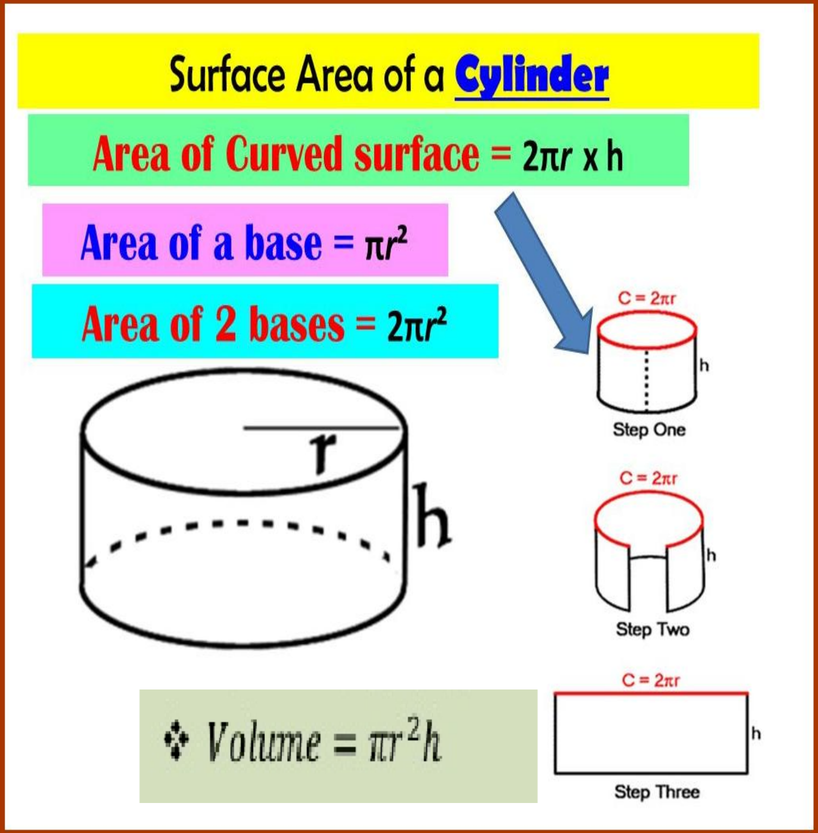
Hence calculating $f(x)$ at $x = 0, \frac{1}{8}, -1, 1$

Value of x	Value of $f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}}$
$x = -1$	$= 18$ ← Maximum
$x = 0$	$f(0) = 12(0)^{\frac{4}{3}} - 6(0)^{\frac{1}{3}} = 0$ ← Minimum
$x = \frac{1}{8}$	$f\left(\frac{1}{8}\right) = 12\left(\frac{1}{8}\right)^{\frac{4}{3}} - 6\left(\frac{1}{8}\right)^{\frac{1}{3}} = \frac{-9}{4}$
$x = 1$	$f(1) = 12(1)^{\frac{4}{3}} - 6(1)^{\frac{1}{3}} = 6$

APPLICATION OF DERIVATIVES-MODULE-9







HOW THE MAXIMA AND MINIMA WORKS IN REAL LIFE SITUATIONS

► **Example 1** A garden is to be laid out in a rectangular area and protected by a chicken wire fence. Show that if only 100 running feet of chicken wire is available for the fence then the area function can be expressed

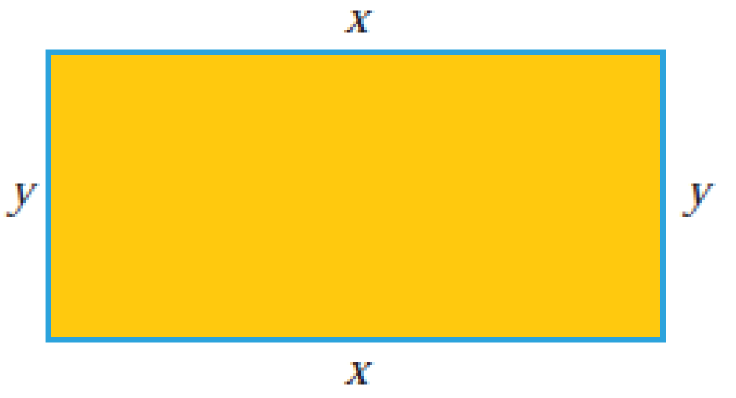
$$A = 50x - x^2$$, where x is length and y is width of the rectangle.

Solution. Let

x = length of the rectangle (ft)

y = width of the rectangle (ft)

A = area of the rectangle (ft^2)



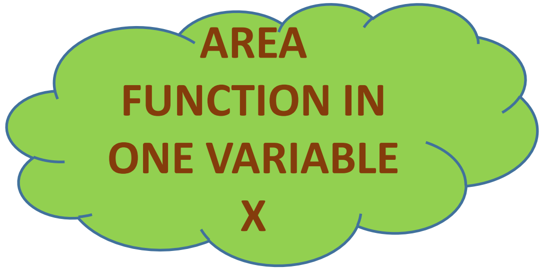
Then

$$A = xy \qquad (1)$$

Since the perimeter of the rectangle is 100 ft, the variables x and y are related by the equation

$$P = 2x + 2y = 100 \quad \text{or} \quad y = 50 - x \qquad (2)$$

(See Figure) Substituting (2) in (1) yields



$$A = x(50 - x) = 50x - x^2 \qquad (3)$$

► **Example 1** A garden is to be laid out in a rectangular area and protected by a chicken wire fence. Show that if only 100 running feet of chicken wire is available for the fence

What would be the length and breadth of the garden, to get maximum area?

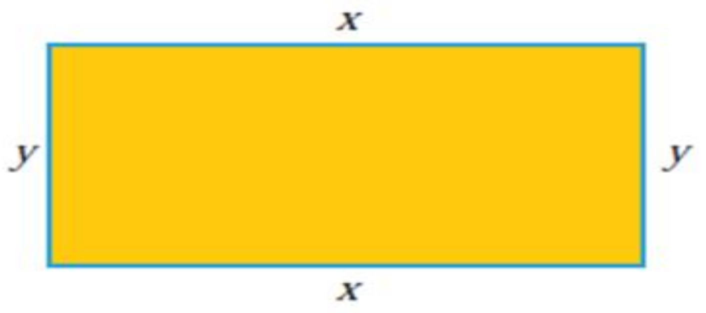
$A = 50x - x^2$, where x is length and y is width of the rectangle.

Solution. Let

x = length of the rectangle (ft)

y = width of the rectangle (ft)

A = area of the rectangle (ft^2)



Then

$$A = xy \quad (1)$$

Since the perimeter of the rectangle is 100 ft, the variables x and y are related by the equation

$$P = 2x + 2y = 100 \quad \text{or} \quad y = 50 - x \quad (2)$$

(See Figure) Substituting (2) in (1) yields

$$A = x(50 - x) = 50x - x^2 \quad (3)$$

50 - 2x = 0 , gives x = 25 (critical point)

A''(x) = -2 < 0 at x = 25

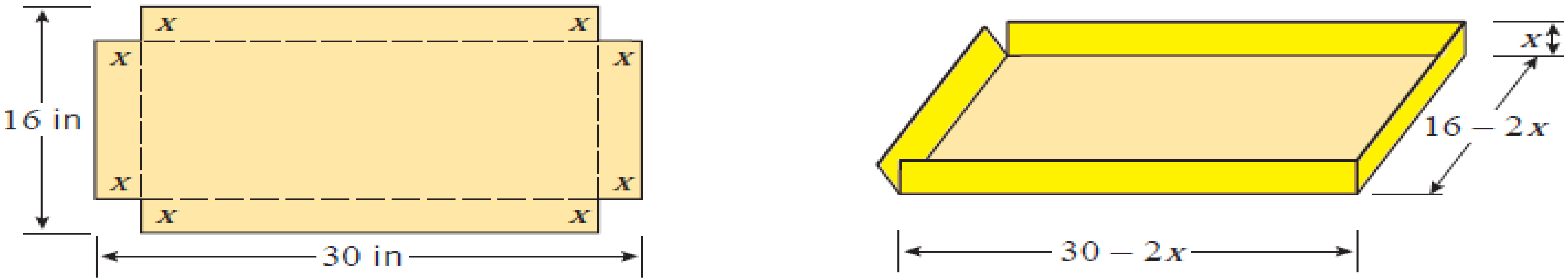
By second derivative test , Area is maximum when x = 25.

Since y = 50 - x , y = 25. Hence,to get maximum area length and breadth should be equal. ie., to get maximum area , this rectangle should be a square with side 25 feet

► **Example 2** An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. Show that the volume function is $V(x) = 480x - 92x^2 + 4x^3$

Solution.

(Figure)



- *Step 1:* Figure illustrates the cardboard piece with squares removed from its corners. Let

$$x = \text{length (in inches) of the sides of the squares to be cut out}$$
$$V = \text{volume (in cubic inches) of the resulting box}$$

- *Step 2:* Because we are removing a square of side x from each corner, the resulting box will have dimensions $16 - 2x$ by $30 - 2x$ by x (Figure 4.5.3b). Since the volume of a box is the product of its dimensions, we have

$$V = (16 - 2x)(30 - 2x)x = 480x - 92x^2 + 4x^3 \qquad (5)$$

Question .

A square piece of tin od side 18 cm is to made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Solution

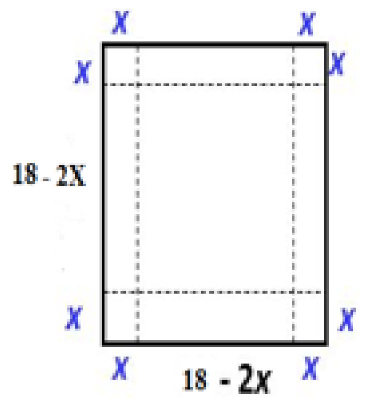
Let the side of the square to be cut off be x cm.Then, the length and the breath of the box will be $(18-2x)$ cm each and the height of the box is x cm

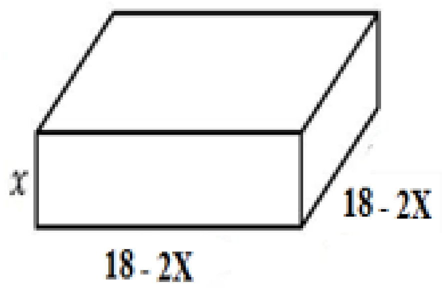
Therefore, the volume V(x) of the box is given by,

$$V(x) = x(18-2x)^2$$
$$\therefore V'(x) = (18-2x)^2 - 4x(18-2x)$$
$$= (18-2x)[18-2x-4x]$$
$$= (18-2x)[18-6x]$$
$$= 6\times 2(9-x)(3-x)$$
$$= 12(9-x)(3-x)$$

For max, V'(x) = 0

$$v'(x) = 0 \Rightarrow x = 9 \text{ or } x = 3$$





$$V''(x) = -12(9 - x + 3 - x)$$
$$= -12(12 - 2x)$$
$$= -24(6 - x)$$

Now, $v'(x) = 0 \Rightarrow x = 9$ or $x = 3$

If x = 9, then the length and the breadth will become 0.

$\therefore x \neq 9$

$\Rightarrow x = 3$

Now, $V''(3) = -24(6-3) = -72 < 0$

$\therefore$ By second derivative test, x = 3 is the point of maxima of V.

Hence, if we remove a square of side 3 cm from each corner of the square tin and make a box from the remaining sheet, then the volume of the box obtained is the largest possible.

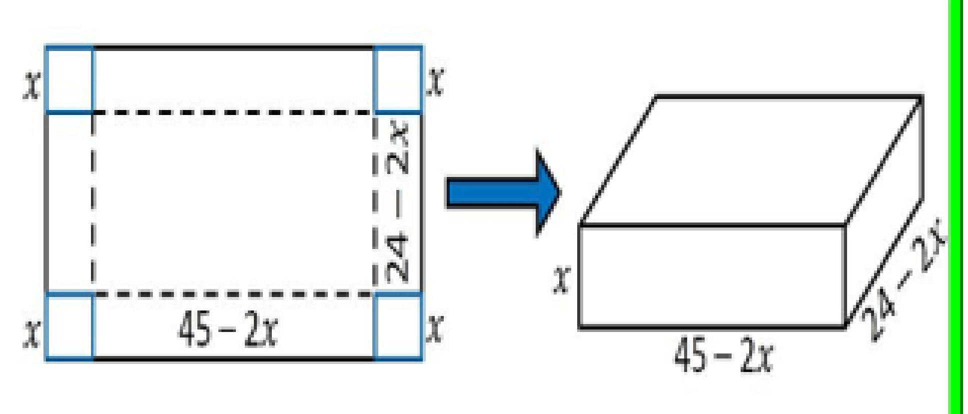
Question 18:

A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Solution 18:

Let the side of the square to be cut be x cm. Then, the height of the box is x, the length is $45 - 2x$, ad the breadth is $24 - 2x$

Therefore, the volume $V(x)$ of the box is given by,



$V(x) = x(45 - 2x)(24 - 2x)$

$= x(1080 - 90x - 48x + 4x^2)$

$= 4x^3 - 138x^2 + 1080x$

$V'(x) = 12x^2 - 276 + 1080$

$= 12(x^2 - 23x + 90)$

$= 12(x - 18)(x - 5)$

For max Volume $V'(x) = 0 \Rightarrow x = 18$ and $x = 5$

$V''(x) = 24x - 276 = 12(2x - 23)$

It is not possible to cut off a square of side 18 cm from each corner of the rectangular sheet. Thus x cannot be equal to 18.

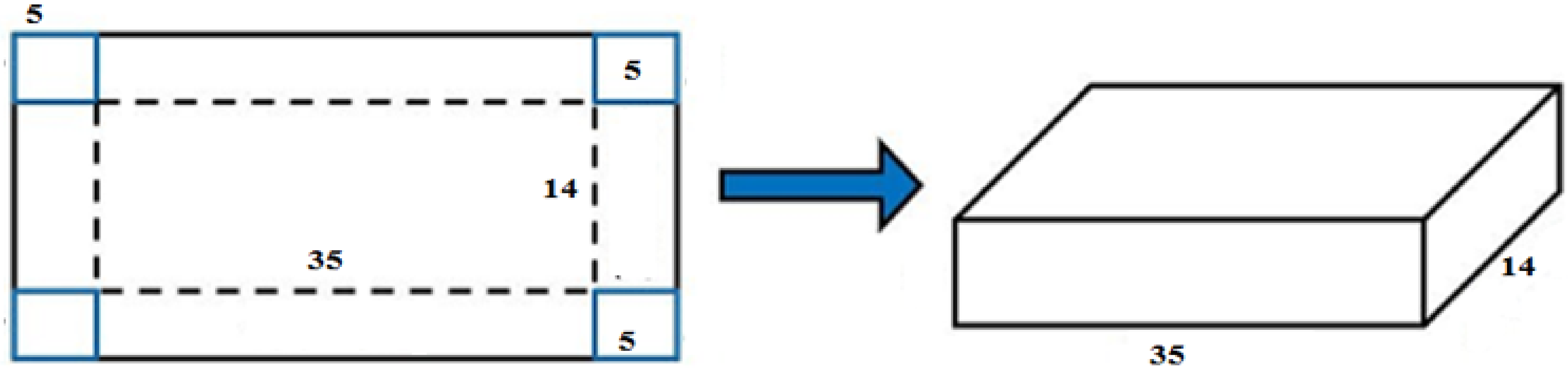
$x = 5$

x = 5

Now, $V''(5) = 12(10 - 23) = 12(-13) = -156 < 0$

∴ By second derivative test x = 5 is the point of maxima.

Hence, the side of the square to be cut off to make the volume of the box maximum possible is 5 cm.



HOME ASSIGNMENT

- EX 6.5
- Q 5 (I), (II),(III), (IV)
- Q6 , Q 7 , Q 8